THE GROUP PROPERTY OF THE INVARIANT S
OF VON NEUMANN ALGEBRAS

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Abstract.

We prove that if M is any countably decomposable factor, the invariant S (M) defined in [1] is a closed subgroup of the group of positive real numbers.
Moreover multiplication by any element of S (M) leaves the spectrum of any state on M invariant.

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Theorem 1 a) Let M be a countably decomposable factor, then the non zero elements of the intersection $S(M)$ of the spectra of the modular operators Δ_φ associated with φ, when φ runs through all faithful normal states on M, is a closed subgroup of the multiplicative group of positive real numbers.
b) For any faithful normal state φ on M the spectrum of Δ_φ is invariant under multiplication by $S(M)$.

To prove the theorem we need a few lemmas.

Let $\mathfrak{A}$ be an achieved generalized left Hilbert algebra, Δ the modular operator of $\mathfrak{A}$.

Lemma 2. Let V be any compact interval of $]0,\infty[$ and χ the characteristic function of V. If $\xi \in \mathfrak{A}$ such that $\chi(\Delta)\xi = \xi$ then for all integers $n \in \mathbb{Z}$ we have $\xi \in \mathcal{D}(\Delta^n)$ and $\Delta^n \xi \in \mathfrak{A}$.

Proof: It is not hard to see that there exists a function $f \in L_1(\mathbb{R})$ such that $\lambda^n = \int_{-\infty}^{+\infty} \lambda^{it} f(t)\, dt$ for all $\lambda \in V$. It then follows that

$$\Delta^n \xi = (\Delta \chi(\Delta))^n \xi = \int_{-\infty}^{+\infty} (\Delta \chi(\Delta))^{it} \xi\, f(t)\, dt$$

$$= \int_{-\infty}^{+\infty} \Delta^{it} \xi\, f(t)\, dt .$$

Clearly $\Delta^n \xi \in \mathcal{D}(\Delta^{1/2})$ and $\| \pi(\Delta^n \xi)\| \leq \|f\|_1$ so that $\Delta^n \xi \in \mathfrak{A}$.

Lemma 3. Let V_1 and V_2 be two compact intervals of $]0,\infty[$ and $V = \{ pq \mid p \in V_1, q \in V_2 \}$.
Let χ_1, χ_2 and χ be the characteristic functions of respectively V_1, V_2 and V. Then for any $\xi_1 \in \mathfrak{A}$, $\xi_2 \in \mathfrak{A}$ such that $\chi_1(\Delta)\xi_1 = \xi_1$ and $\chi_2(\Delta)\xi_2 = \xi_2$ we have $\chi(\Delta)\xi_1\xi_2 = \xi_1\xi_2$.

<u>Proof:</u> By lemma 2 we know that $\Delta^n \xi_1 \in \mathfrak{A}$ for all $n \in \mathbb{Z}$. With the notations of [2] and using [2, lemma 8.3] this implies that $\Delta^n \xi_1 \in \mathfrak{A}'$ for all $n \in \mathbb{Z}$ and therefore $\Delta^n \xi_1 \in \mathfrak{A}^{\#}$. This holds also for $\Delta^n \xi_2$ and by induction we get that $\xi_1 \xi_2 \in \mathcal{D}(\Delta^n)$ and that $\Delta^n(\xi_1 \xi_2) = (\Delta^n \xi_1)(\Delta^n \xi_2)$.
If we put $\Delta_1 = \Delta \chi_1(\Delta)$, then $Sp\ \Delta_1 \subset V_1$ and $\Delta^n \xi_1 = \Delta_1^n \xi_1$. For any simply closed smooth curve Γ enclosing V_1 we have

$$\Delta^n(\xi_1 \xi_2) = \pi'(\Delta^n \xi_2)\, \Delta_1^n \xi_1$$

$$= \frac{i}{2\pi} \oint_\Gamma \pi'((\lambda\Delta)^n \xi_2)(\Delta_1 - \lambda)^{-1} \xi_1 \, d\lambda$$

As in the proof of lemma 2 we can find a function $f \in L_1(\mathbb{R})$ such that $(\Delta_1 - \lambda)^{-1} \xi_1 = \int_{-\infty}^{+\infty} \Delta^{it} \xi_1 \, f(t)\, dt$ and by the same arguments $(\Delta_1 - \lambda)^{-1} \xi_1 \in \mathfrak{A}$ whenever $\lambda \notin V_1$.
So for any polynomial p we have

$$p(\Delta)\, \xi_1 \xi_2 = \frac{i}{2\pi} \oint_\Gamma ((\Delta_1 - \lambda)^{-1} \xi_1)\; p(\lambda\Delta)\, \xi_2 \, d\lambda .$$

Now let V_o be any compact interval disjoint from V and $E_o = \chi_o(\Delta)$ where χ_o is the characteristic function of V_o. Then

$$p(\Delta) E_o \xi_1 \xi_2 = p(\Delta E_o)\, \xi_1 \xi_2$$

$$= \frac{i}{2\pi} \oint_\Gamma E_o((\Delta_1 - \lambda)^{-1} \xi_1)\; p(\lambda\Delta)\, \xi_2 \, d\lambda$$

Choose ε sufficiently small such that the two open sets

$$W_o = \{ z \mid z \in \mathbb{C} , \text{ distance } (z, V_o) < \varepsilon \}$$

$$W = \{ z \mid z \in \mathbb{C} , \text{ distance } (z, V) < \varepsilon \}$$

have disjoint closures.

Then it is possible to choose Γ such that the set $\{ pq \mid p \in V_2 , q$ is inside $\Gamma \}$ is contained in W. Let f be the analytic function on $W_o \cup W$ which is 1 on W_o and 0 on W. By Runge's theorem it is possible to find a sequence of polynomials p_k such that p_k tends uniformly to f on $W_o \cup W$. Then $p_k(\Delta E_o)\, \xi_1 \xi_2$ tends to $E_o \xi_1 \xi_2$ and $p_k(\lambda \Delta \chi_2(\Delta))\, \xi_2$ tends to 0 uniformly in $\lambda \in \Gamma$. Moreover $\| \pi((\Delta_1 - \lambda)^{-1} \xi_1)\|$ is uniformly bounded on Γ. Therefore $E_o \xi_1 \xi_2 = 0$ and since this holds for all compact closed intervals disjoint from V, $\chi(\Delta)\, \xi_1 \xi_2 = \xi_1 \xi_2$. This completes the proof.

Let φ be a faithful normal state on the von Neumann algebra M. Let $(M, \mathcal{H}, \xi_o)$ be the G.N.S.-construction of φ on M. As in [2] let $S = J\Delta^{1/2}$ be the corresponding involution. Remind that JMJ = M', and that $\sigma_t(x) = \Delta^{it} x \Delta^{-it}$ for $x \in M$ defines a one parameter group of automorphisms of M. In [2, lemma 15.8] it is proved that the subalgebra $\{ x \in M \mid \sigma_t(x) = x$ for all $t \in \mathbb{R} \}$ equals the set $\{ x \in M \mid \varphi(xy) = \varphi(yx)$ for all $y \in M \}$. As in [2] we call this subalgebra M_φ.

Let e be a non zero projection of M_φ, we shall first determine the modular operator of the state φ_e defined on the reduced von Neumann algebra M_e by $\varphi_e(x) = \varphi(x) / \varphi(e)$. The closed subspace $\mathcal{H}_e$ = Image $e \cap$ Image $JeJ = eJeJ\mathcal{H}$ is invariant by any element of the algebra M_e. So we can consider the algebra M_1 induced by M_e in $\mathcal{H}_e$ and the canonical homomorphism π of M_e onto M_1. The element $e\xi_o$ of $\mathcal{H}$ is in $\mathcal{H}_e$ because $JeJ\xi_o = Je\xi_o = Je\Delta^{1/2}\xi_o = J\Delta^{1/2}e\xi_o = e\xi_o$ hence $e\xi_o \in eJeJ\mathcal{H}$. Let $\xi_1 = e\xi_o / \|e\xi_o\|$, then it is easy to check that $(\pi, \mathcal{H}_e, \xi_1)$ is the G.N.S.-construction of the state φ_e on M_e. To check that ξ_1 is cyclic for M_1 in $\mathcal{H}_e$ it is enough to prove that $x \in M$ implies $eJeJx\xi_o \in M_1 \xi_1$ which follows from the equality $eJeJx\xi_o = exJeJ\xi_o = exe\xi_o$.

Now $e\Delta^{it} = \Delta^{it} e$ for all $t \in \mathbb{R}$ and similarly JeJ commutes with Δ^{it} for all t, so Δ leaves $\mathcal{H}_e$ invariant and its restriction to $\mathcal{H}_e$ is a closed positive operator.

Let $x \in M_1$, then there exists an X in M_e such that $\pi(X) = x$, in particular $\| e\xi_0 \| \, x\xi_1 = xe\xi_0 = X\xi_0$ and $\| e\xi_0 \| \, x^*\xi_1 = x^* e\xi_0 = X^*\xi_0$, hence $Sx\xi_1 = x^*\xi_1$ and the involution S_e corresponding to $(M_1, \mathcal{H}_e, \xi_1)$ coincides with S on $M_1\xi_1$. Similarly we get the coincidence of F_e with F on $M_1'\xi_1$. It follows that $S_e \doteq J_R \Delta_R^{1/2}$ where J_R is the restriction of J to $\mathcal{H}_e$ and Δ_R the restriction of Δ to $\mathcal{H}_e$. By the uniqueness of the polar decomposition of closed operators we get the equality $\Delta_e = \Delta_R$. Hence the modular operator of the state φ_e on M_e is the restriction of the modular operator of φ on M to the invariant subspace $eJeJ\mathcal{H}$.

<u>Definition 4.</u> For a faithful normal state φ on M put $\Gamma_\varphi = \bigcap$ spectrum of the modular operator of φ_e on M_e when e runs through all non zero projections of the center of M_φ.

<u>Lemma 5.</u> Let $\lambda_1 > 0$, $\lambda_1 \in \Gamma_\varphi$ and $\lambda_2 > 0$, $\lambda_2 \in Sp\,\Delta$ then $\lambda_1\lambda_2 \in Sp\,\Delta$.

<u>Proof:</u> a) We first show that if a bounded open interval V of $]0,\infty[$ intersects $Sp\,\Delta$ there exists a non zero $x \in M$ with $\chi(\Delta)x\xi_0 = x\xi_0$, χ being the characteristic function of V. By hypothesis $\chi(\Delta) \neq 0$, so there is a $y \in M$ with $\chi(\Delta)y\xi_0 \neq 0$. Let χ_n be a sequence of C^∞ functions on $]0,\infty[$ with $0 \leq \chi_n \leq \chi$ and $\chi_n(\Delta) \to \chi(\Delta)$ strongly when $n \to \infty$. Then there exists an n with $\chi_n(\Delta)y\xi_0 \neq 0$, by [2] one has $\chi_n(\Delta)y\xi_0 \in M\xi_0$, and obviously $\chi(\Delta)\chi_n(\Delta)y\xi_0 = \chi_n(\Delta)y\xi_0$.

b) Let V_1 be a compact interval of $]0,\infty[$ with λ_1 in its interior, then let e be a non zero projection of the center of M_φ.

Since the interior of V_1 intersects $Sp\ \Delta_e$ there exists by a) an element $x \neq o$ of the reduced induced algebra M_1 of M in $eJeJ\mathcal{H}$ such that $x\xi_1 = \chi_1(\Delta_e)x\xi_1$ where χ_1 is the characteristic function of V_1. Now $x\xi_1 \in \mathcal{H}_e$, hence $\chi_1(\Delta_e)x\xi_1 = \chi_1(\Delta)x\xi_1$. Since $x \in M_1$ there exists an X in M_e with $x\xi_1 = X\xi_0$, so $\chi_1(\Delta)X\xi_0 = X\xi_0$, $X \neq o$, X in M_e.

We claim that for such V_1 the supremum $\bigvee$ Supp x, where x runs over all elements in M with $\chi_1(\Delta)x\xi_0 = x\xi_0$, is equal to one. In fact it is a certain projection k with for all $t \in \mathbb{R}$, $\Delta^{it}k\Delta^{-it} = k$ because $\chi_1(\Delta)\Delta^{it}x\Delta^{-it}\xi_0 = \Delta^{it}\chi_1(\Delta)x\xi_0$.
Also for all unitary $u \in M_\varphi$, $uku^* = k$ because $\chi_1(\Delta)uxu^*\xi_0 = \chi_1(\Delta)uJuJx\xi_0 = uJuJ\chi_1(\Delta)x\xi_0$ since u and J u J commute with Δ. So we know that k belongs to the center of M_φ hence 1 - k is a projection e in the center of M_φ. If $e \neq o$, there exists an $X \in M$ with $X\xi_0 = \chi_1(\Delta)X\xi_0$, $X \neq o$, $eX = Xe = X$, so Supp $X \leq e$ which contradicts Supp $X \leq k$ if $X \neq o$.

c) Now let W be any neighbourhood of $\lambda_1\lambda_2$ in $]o, \infty[$, choose V_1 and V_2 compact intervals containing respectively λ_1 and λ_2 in their interior and such that $V_1 \cdot V_2 \subset W$. Let χ_1, χ_2 and χ be the respective characteristic functions of V_1, V_2 and V. By a) there exists $x \in M$ with $x \neq o$ and $x\xi_0 = \chi_2(\Delta)x\xi_0$, by b) there exists $y \in M$ with $y\xi_0 = \chi_1(\Delta)y\xi_0$ and $yx \neq o$ because $1 = \bigvee$ Supp y, when y runs over all elements in M satisfying $\chi_1(\Delta)y\xi_0 = y\xi_0$. If we apply lemma 3 to the left generalised Hilbert algebra $\mathfrak{A} = M\xi_0$ we get $\chi(\Delta)yx\xi_0 = yx\xi_0$ hence V intersects the spectrum of Δ. It then follows that $\lambda_1\lambda_2 \in Sp\ \Delta$ as far as W was arbitrary.

<u>Proof of the theorem.</u> Since the theorem is obvious in the semi-finite case we assume M is type III. It is enough to prove b). Let φ be a faithful normal state on M, let $\lambda_2 > o$, $\lambda_2 \in Sp\ \Delta_\varphi$, let

$\lambda_1 > 0$, $\lambda_1 \in S(M)$, then $\lambda_1 \lambda_2 \in Sp\, \Delta_\varphi$ will follow from the inclusion $S(M) \subset \Gamma_\varphi$. This inclusion is true because for each non zero projection e in the center of M_φ , M_e is isomorphic to M and hence $Sp\, \Delta_{\varphi_e} \supset S(M)$ because φ_e is a faithful normal state on M_e.

This result will be used later to improve the classification of type III factors.

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References.

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